Last Time: Uniqueness of RREF

WTS

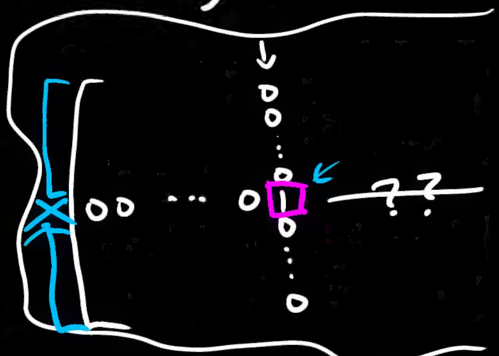
Thm: RREF's are uniquely determined.

we've shown (up to now):

1. Elementary row ops are "reversible"
   ↳ "row equivalence" is an equivalence relation.
2. Linear Combination Lemma
   ↳ If A row-reduces to B, then rows of A are lin. comb. of rows of B

Lem: If M is in RREF, then nonzero rows of M are not linear combinations of the other rows.

pf: Let M be a matrix in RREF. Every nonzero row of M has a leading 1. Furthermore, all leading 1's are the only nonzero entries in their column.



In particular, every linear combination of the other rows has 0 in the column corresponding to any given leading 1; hence that row is not a lin. comb. of the other rows (they don't match in that coord!) $\boxtimes$

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pf (Uniqueness of RREF): Let M be a matrix with m rows. We proceed by induction on the number of columns of M.

Base Case: If M has only 1 column, either all entries of this column are 0 or not. If all entries of the column are 0, then M is in RREF. Otherwise, this column has a nonzero entry. Swap any such entry to the first position, multiply by a suitable nonzero scalar, and finally eliminate all other entries. The result is an $m \times 1$ matrix with 1 in the first entry and 0's in all other entries. Hence M has a unique RREF in these cases.

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{all rows are zero!}$$

$$k \neq 0 \quad \begin{bmatrix} \vdots \\ k \\ \vdots \end{bmatrix} \rightsquigarrow \begin{bmatrix} k \\ \vdots \\ \\ \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ \vdots \\ \\ \end{bmatrix}$$

$$\rightsquigarrow \begin{array}{l} r_1 \to \\ \\ r_i \to \end{array} \begin{bmatrix} 1 \\ \vdots \\ a \\ \vdots \end{bmatrix} \xrightarrow{r_i - a r_1} \begin{bmatrix} 1 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Induction Step: Suppose M has $n+1$ columns and suppose every $m \times n$ matrix has a unique RREF. Suppose M has two RREFs, B and C. Because A is an $m \times n$ matrix, our assumption yields B and C have the same first n columns (because our RREFs for M contain an RREF for A).

$$M = [\underbrace{A}_{n \text{ columns}} \mid \vec{a}]$$

$$B = [\operatorname{rref}(A) \mid \vec{b}]$$

$$C = [\operatorname{rref}(A) \mid \vec{c}]$$

Consider the homogeneous linear systems determined by B and C (i.e. $B\vec{x} = \vec{0}$ and $C\vec{x} = \vec{0}$). If $B \neq C$, they differ in the last column, so

we could find a row i so that $\underbrace{b_i \neq c_i}$ (where $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ and $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$). Either row i has a leading 1 in rref (A) or it is an all-zeros row for rref(A). We may subtract row i of B from row i of C. In the corresponding linear systems, we obtain the equation $(c_i - b_i)x_n = 0$. Thus either $\underbrace{c_i - b_i = 0}$ or $x_n = 0$. As $b_i \neq c_i$, we must have $[x_n = 0]$ in the solution of this linear system, thus row i must have a leading 1 in column n (b/c x_n is not a free variable). Hence there is exactly one entry in column n which is nonzero. This leading 1 must occur in exactly the same position in both B and C because of the RREF ordering on rows w/ leading 1's. Hence $B = C$ is the unique RREF for M (which is what we wanted ☺). ▨

Point: Every matrix is row-equivalent to a unique matrix in RREF.

Cor: A matrix A and matrix B are row-equivalent if and only if $\text{rref}(A) = \text{rref}(B)$.

Ex: Which of these matrices are row-equivalent?

$$A=\begin{bmatrix}1&3\\2&4\end{bmatrix}\quad B=\begin{bmatrix}1&5\\2&10\end{bmatrix}\quad C=\begin{bmatrix}1&-1\\3&0\end{bmatrix}$$

$$D=\begin{bmatrix}2&6\\4&10\end{bmatrix}\quad E=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\quad F=\begin{bmatrix}3&3\\2&2\end{bmatrix}$$

Sol: Compute RREF for each:

$$A:\ \begin{bmatrix}1&3\\2&4\end{bmatrix}\xrightarrow{\ell_2-2\ell_1}\begin{bmatrix}1&3\\0&-2\end{bmatrix}\xrightarrow{-\frac{1}{2}\ell_2}\begin{bmatrix}1&3\\0&1\end{bmatrix}\xrightarrow{\ell_1-3\ell_2}\underbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}\overset{\text{rref}(A)}{=}$$

$$B:\ \begin{bmatrix}1&5\\2&10\end{bmatrix}\xrightarrow{\ell_2-2\ell_1}\begin{bmatrix}1&5\\0&0\end{bmatrix}=\text{rref}(B)$$

$$C:\ \begin{bmatrix}1&-1\\3&0\end{bmatrix}\xrightarrow{\frac{1}{3}\ell_2}\begin{bmatrix}1&-1\\1&0\end{bmatrix}\xrightarrow{\ell_2-\ell_1}\begin{bmatrix}1&-1\\0&1\end{bmatrix}\xrightarrow{\ell_1+\ell_2}\underbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}=\text{rref}(C)$$

$$D:\ \begin{bmatrix}2&6\\4&10\end{bmatrix}\xrightarrow{\ell_2-2\ell_1}\begin{bmatrix}2&6\\0&-2\end{bmatrix}\xrightarrow[-\frac{1}{2}\ell_2]{\frac{1}{2}\ell_1}\begin{bmatrix}1&3\\0&1\end{bmatrix}\xrightarrow{\ell_1-3\ell_2}\underbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}=\text{rref}(D)$$

$$E:\ \begin{bmatrix}0&1\\-1&0\end{bmatrix}\xrightarrow{-\ell_2}\begin{bmatrix}0&1\\1&0\end{bmatrix}\xrightarrow{\ell_1\leftrightarrow\ell_2}\underbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}=\text{rref}(E)$$

$$F:\ \begin{bmatrix}3&3\\2&2\end{bmatrix}\xrightarrow[\frac{1}{2}\ell_2]{\frac{1}{3}\ell_1}\begin{bmatrix}1&1\\1&1\end{bmatrix}\xrightarrow{\ell_2-\ell_1}\begin{bmatrix}1&1\\0&0\end{bmatrix}=\text{rref}(F)$$

We have $\text{rref}(A)=\text{rref}(C)=\text{rref}(D)=\text{rref}(E)=\begin{bmatrix}1&0\\0&1\end{bmatrix}$
So A, C, D, E are row equivalent.

OTOH, $\text{rref}(B)=\begin{bmatrix}1&5\\0&0\end{bmatrix}$ and $\text{rref}(F)=\begin{bmatrix}1&1\\0&0\end{bmatrix}$, so these are inequivalent to the others on our list. $\boxtimes$

weakly related: $m\Big[\ \overset{n}{A}\ \Big]\vec{x}=\vec{0}$ If $m<n$, then this system has infinitely many solutions.

Ex: Write down all possible 2×3 linear systems (homogeneous) up to row equivalence.

Sol: We give all RREF 2×3 matrices below. for $a,b,c \in \mathbb{R}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}, \begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, every homogeneous 2×3 linear system has the same solution set as $A\vec{x} = \vec{0}$ for one of the matrices A listed above. $\boxtimes$

Linear Maps (determined by matrices)

Def^n: A function $L: \mathbb{R}^n \to \mathbb{R}^m$ is linear when $L(\vec{u} + a\vec{v}) = L(\vec{u}) + aL(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $a \in \mathbb{R}$.

Ex: $L: \mathbb{R}^2 \to \mathbb{R}$ defined by $L\begin{bmatrix} x \\ y \end{bmatrix} = x + y$ is a linear map. Indeed, given $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ and $c \in \mathbb{R}$, we have:

$$L\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + a\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = L\begin{bmatrix} x_1 + ax_2 \\ y_1 + ay_2 \end{bmatrix} = (x_1 + ax_2) + (y_1 + ay_2)$$

$$= (x_1 + y_1) + a(x_2 + y_2)$$

$$= L\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + aL\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}. \qquad \boxtimes$$

__Non-ex:__ $L : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $L[x] = [x^2]$ is not a linear map. To show this, we must find $[x], [y] \in \mathbb{R}^1$ and $a \in \mathbb{R}$ s.t. $L([x] + a[y]) \neq L[x] + aL[y]$.

Trying $a = x = y = 1$, we see

$L([1] + 1[1]) = L[2] = [4]$ whereas

$$\not\parallel$$

$L[1] + 1L[1] = [1] + [1] = [2]$

So we've verified L is not linear... $\boxtimes$